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Rasanen, Syksy

2014-03

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Rasanen , S 2014 , ' A covariant treatment of cosmic parallax ' , Journal of Cosmology and  
Astroparticle Physics , vol. 2014 , no. 3 , 035 . <https://doi.org/10.1088/1475-7516/2014/03/035>

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<http://hdl.handle.net/10138/310433>

<https://doi.org/10.1088/1475-7516/2014/03/035>

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# A covariant treatment of cosmic parallax

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**Abstract.** The Gaia satellite will soon probe parallax on cosmological distances. Using the covariant formalism and considering the angle between a pair of sources, we find parallax for both spacelike and timelike separation between observation points. Our analysis includes both intrinsic parallax and parallax due to observer motion. We propose a consistency condition that tests the FRW metric using the parallax distance and the angular diameter distance. This test is purely kinematic and relies only on geometrical optics, it is independent of matter content and its relation to the spacetime geometry. We study perturbations around the FRW model, and find that they should be taken into account when analysing observations to determine the parallax distance.

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## 1 Introduction

**Measuring cosmic parallax.** Parallax, i.e. change in the angular position of objects on the sky when viewed from different locations, has had important implications for cosmology from the debate on the heliocentric system in the 16th century to the modern distance ladder. Parallax is an attractive way to measure distances, because in contrast to angular diameter or luminosity, no information about source properties is needed, as the reference quantity is the distance between observation points, which is directly measured. However, so far distance measurements using parallax have been restricted to nearby objects in the Galaxy. The longest parallax distance currently measured, by the Hipparcos satellite<sup>1</sup>, is of the order 100 pc.

The possibility of measuring parallax over cosmological distances, cosmic parallax, was first raised in 1935 by McCrea [1], though he considered it impossible in practice. If only the difference in the position of the Earth with respect to the Sun is considered, the maximum spatial separation between observation points, called the baseline, is two astronomical units (AU). In the 1970s it was suggested that advances in spacecraft technology would make cosmic parallax measurements feasible in the near future [2–6]. In 1986 Kardashëv proposed using the distance travelled by the Earth with respect to the cosmic microwave background (CMB) as a baseline, instead of the difference in position relative to the Sun [7]. The Earth moves with respect to the CMB with a velocity of 369 km/s (with an annual modulation

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<sup>1</sup><http://sci.esa.int/hipparcos>

of 30 km/s due to motion around the Sun) [8]<sup>2</sup>, so the change in position during one year is 78 AU. Furthermore, this baseline increase is secular, unlike that due to Earth's motion around the Sun. Because primordial cosmological perturbations are close to adiabatic [9], the average rest frame of distant sources, i.e. the frame of statistical homogeneity and isotropy of the matter distribution, is close to the rest frame of the CMB.

Cosmic parallax due to motion with respect to the CMB will soon be probed for the first time by the Gaia satellite<sup>3</sup>, which was launched on December 19, 2013. Over a period of five years, Gaia is expected to measure the angular positions of about 500 000 quasars and 3 million galaxies at cosmological distances with a precision of order 100  $\mu$ as [10–17].

**Cosmic parallax as a test of homogeneity and isotropy.** The angular diameter distance  $D_A$  and the luminosity distance  $D_L$  are related trivially by the Etherington relation  $D_L = (1 + z)^2 D_A$  [18, 19], but the parallax distance  $D_P$  contains independent information. McCrea pointed out already in 1935 that this makes it possible to test the FRW metric by comparing measurements of  $D_P$  and measurements of  $D_A$  (or  $D_L$ ) [1]. In [2] the more limited point was made that if the universe is described by the FRW metric, measuring both  $D_P$  and  $D_L$  makes it possible to determine the metric fully, i.e. to find both the scale factor  $a(t)$  and the curvature constant  $K$ . The parallax distance was given for an arbitrary spacetime with zero null shear in [20] and properly derived in [21], and derived for the general case in [22]. Like the original work of McCrea, these papers studied the angular position of a single source relative to a constant baseline direction between two observation points separated by a spacelike interval. We refer to this setup as the 'classic' parallax case.

The classic analysis is not straightforwardly applicable to real observations, because in them the interval between observation points is timelike, not spacelike. Also, in practice positions of sources on the sky are measured relative to each other, instead of comparing a single source to a constant baseline direction. In 1988, Hasse and Perlick derived the general condition under which relative positions of sources on the sky remain constant, assuming that the world lines of observers and sources are integral curves of the same timelike vector field [23] (see also [24]). In this case, parallax of all pairs of sources vanishes for all observers if and only if the vector field is conformally stationary. (In the case of observers comoving with dust matter, this means that the spacetime is either FRW or stationary or both.) This is the same as the condition for all observers to measure vanishing CMB anisotropy [24–26]. The CMB conditions are stable in one direction, but not in the other. If the spacetime is close to FRW, the CMB is almost isotropic. However, small CMB anisotropy does not imply that the spacetime would be close to FRW (or stationary), even when the matter is dust [27–30]. There has been no similar analysis of the stability of the parallax conditions. If the universe is close to FRW, is parallax necessarily small? If parallax is small for all observers, does that imply that the universe is close to FRW? Also, the analysis in [23, 24] does not consider a difference between the velocity field of observers and the velocity field of sources, which is central in the parallax due to our motion with respect to the frame of homogeneity and isotropy.

Using the parallax to test for deviations from the FRW metric has come up recently in connection with models where a large spherical inhomogeneity has been suggested as an alternative to dark energy [14, 15, 31, 32], as well as proposed models with large-scale

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<sup>2</sup>Assuming that the almost all of the CMB dipole is due to a Doppler effect. Without this assumption, the velocity is determined to be  $(384 \pm 139)$  km/s [8].

<sup>3</sup><http://www.cosmos.esa.int/web/gaia>

anisotropy [15, 32, 33]. The idea is that if the universe is on large scales well described by the FRW metric, the parallax is small, so any detection indicates deviation from the FRW case. The parallax due to our motion with respect to the frame of statistical homogeneity and isotropy has been considered noise, and cosmic parallax has been presented as being due to anisotropic expansion, i.e. shear, and characterised as a property of spacetime, as opposed to parallax due to observer motion. However, while all models with shear have non-vanishing cosmic parallax, there are shear-free models with cosmic parallax [34]. In any case, shear is a property of a velocity field, and the spacetime geometry constrains the velocity field, but does not determine it. The separation between parallax due to our motion and parallax as a property of spacetime depends on the choice of frame. It has also been proposed that instead of treating the parallax due to our motion as noise, it could be used to put constraints on dark energy models, assuming that the universe is described by the FRW metric [13]. These studies are model-specific, and have all used classic parallax results based on spacelike separation. However, it is possible to use the parallax distance for a kinematic test of the FRW metric that relies only on geometrical optics, and is independent of the matter content or the dynamics that relates it to the spacetime geometry (usually given by the Einstein equation).

We will expand and complement previous work. In section 2 we derive the parallax for a pair of sources in a general spacetime in terms of covariant quantities. We consider both spacelike and timelike separation, and include both the parallax due to deviation of the spacetime from conformal stationarity and the parallax due to difference between observer and source velocity. We also discuss the general relation between  $D_P$  and  $D_A$ . In section 3 we specialise to the FRW universe. We introduce a consistency condition between  $D_P$  and  $D_A$  that is specific to the FRW metric. In section 4 we consider the perturbed FRW case and study the stability of the FRW results. In section 5 we discuss non-perturbative deviations from the FRW case, different kinds of tests of homogeneity and isotropy and the accuracy with which the parallax distance will be probed by Gaia. In section 6 we summarise our results and mention open issues.

## 2 General spacetime

### 2.1 Geometrical setup

**The two frames.** We mostly follow the notation of [35]; for reviews of the covariant approach, see [19, 26, 36–38]. We consider two timelike velocity fields,  $n$  and  $u$ . Both are normalised to unity,  $n \cdot n = u \cdot u = -1$ , but are otherwise arbitrary (we use the notation  $a \cdot b \equiv g_{\alpha\beta} a^\alpha b^\beta$ , where  $g_{\alpha\beta}$  is the metric, for any vectors  $a$  and  $b$ ). We take  $u$  to be the velocity of the observer, and we will later take  $n$  to correspond to the frame of statistical homogeneity and isotropy. Without loss of generality, we can write  $u$  in terms of  $n$  and a vector  $v$  that is orthogonal to  $n$ ,

$$u^\alpha = \gamma(n^\alpha + v^\alpha) , \quad (2.1)$$

where  $\gamma \equiv -n \cdot u = (1 - |v|^2)^{-1/2}$ , with  $|v|^2 \equiv v \cdot v$  and  $v \cdot n = 0$ . The tensors that project on the rest spaces orthogonal to  $n$  and  $u$  are

$$\begin{aligned} h_{\alpha\beta} &\equiv g_{\alpha\beta} + n_\alpha n_\beta \\ h_{\alpha\beta}^{(u)} &\equiv g_{\alpha\beta} + u_\alpha u_\beta . \end{aligned} \quad (2.2)$$

The derivative of  $u$  can be decomposed as

$$\nabla_\beta u_\alpha = \frac{1}{3} h_{\alpha\beta}^{(u)} \theta^{(u)} + \sigma_{\alpha\beta}^{(u)} + \omega_{\alpha\beta}^{(u)} - \dot{u}_\alpha u_\beta , \quad (2.3)$$

where  $\theta^{(u)} \equiv \nabla_\alpha u^\alpha$  is the volume expansion rate,  $\sigma_{\alpha\beta}^{(u)} \equiv h_{\alpha\gamma}^{(u)} h_{\beta\delta}^{(u)} \nabla^\delta u^\gamma - \frac{1}{3} \theta^{(u)} h_{\alpha\beta}^{(u)}$  is the shear tensor,  $\omega_{\alpha\beta}^{(u)} \equiv \nabla_{[\beta} u_{\alpha]} + \dot{u}_{[\alpha} u_{\beta]}$  is the vorticity tensor, and  $\dot{u}^\alpha \equiv u^\beta \nabla_\beta u^\alpha$  is the acceleration vector. Overdot refers to derivative along  $u$ .

**Light bundles.** In the geometrical optics approximation light travels on null geodesics [39] (page 93). We denote the momentum of a light ray labelled  $A$  by  $k_A$ , and we have  $k_A \cdot k_A = 0$  and  $k_A^\alpha \nabla_\alpha k_A = 0$ . Photon energy measured in the two frames is

$$\begin{aligned} E_A^{(n)} &= -n \cdot k_A \\ E_A^{(u)} &= -u \cdot k_A . \end{aligned} \quad (2.4)$$

The photon momentum can be decomposed as

$$\begin{aligned} k_A^\alpha &= E_A^{(n)} (n^\alpha + e_A^\alpha) \\ &= E_A^{(u)} (u^\alpha + r_A^\alpha) , \end{aligned} \quad (2.5)$$

with  $n \cdot e_A = 0$ ,  $e_A \cdot e_A = 1$  and  $u \cdot r_A = 0$ ,  $r_A \cdot r_A = 1$ . We introduce the 'normalised' dimensionless photon momentum

$$\hat{k}_A^\alpha \equiv E_A^{(u)-1} k_A^\alpha = u^\alpha + r_A^\alpha . \quad (2.6)$$

The relation between the two decompositions of  $k_A$  is illustrated in figure 1.

We define a tensor that projects orthogonally to  $k_A$  as

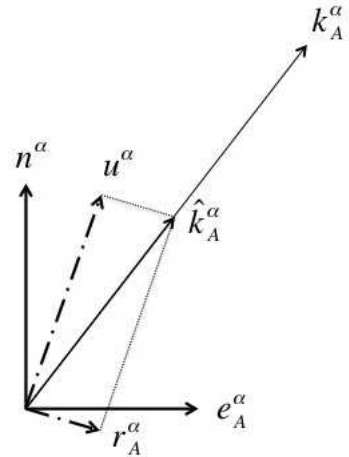
$$\begin{aligned} \tilde{h}_{A\alpha\beta} &= g_{\alpha\beta} + n_\alpha n_\beta - e_{A\alpha} e_{A\beta} \\ &= g_{\alpha\beta} - E_A^{(n)-2} k_{A\alpha} k_{A\beta} + 2E_A^{(n)-1} k_{A(\alpha} n_{\beta)} . \end{aligned} \quad (2.7)$$

Note that  $\tilde{h}_{A\alpha\beta}$  has been chosen to project orthogonally also to  $n$  (and thus  $e_A$ ), whereas  $\hat{k}_A$  is defined with respect to  $u$ . The observer frame is more relevant for observations, but  $n$ , once it is chosen to correspond to the frame of statistical homogeneity and isotropy, is better adapted to the spacetime geometry.

The derivative of  $k_A$  can be decomposed as

$$\nabla_\beta k_{A\alpha} = \frac{1}{2} \tilde{h}_{A\alpha\beta} \tilde{\theta}_A + \tilde{\sigma}_{A\alpha\beta} + k_{A(\alpha} P_{\beta)} , \quad (2.8)$$

where  $\tilde{\theta}_A \equiv \tilde{h}_{A\beta}^\alpha \nabla_\alpha k_A^\beta = \nabla_\alpha k_A^\alpha$  is the expansion rate of the area of a bundle of null geodesics and  $\tilde{\sigma}_{A\alpha\beta} \equiv \tilde{h}_{A\alpha}^\delta \tilde{h}_{A\beta}^\gamma \nabla_\gamma k_{A\delta} - \frac{1}{2} \tilde{h}_{A\alpha\beta} \tilde{\theta}_A$  is the null shear. We have  $\tilde{\sigma}_{A\alpha\beta} k_A^\beta = 0$ ,  $P_A \cdot k_A = 0$ . The null shear scalar is defined as  $\tilde{\sigma}_A \equiv \sqrt{\frac{1}{2} \tilde{\sigma}_{A\alpha\beta} \tilde{\sigma}_A^{\alpha\beta}}$ . The scalars  $\tilde{\theta}_A$  and  $\tilde{\sigma}_A$  do not



**Figure 1.** The relation between  $k_A$ ,  $\hat{k}_A$ ,  $n$ ,  $e_A$ ,  $u$  and  $r_A$ .

depend on the choice of  $\tilde{h}_{A\alpha\beta}$ , as long as it is orthogonal to  $k_A$ . The vector  $P_A$  does depend on this choice; for the tensor (2.7) we have

$$P_A^\alpha = -2E_A^{(n)-1} n^\beta \nabla_\beta k_A^\alpha - E_A^{(n)-2} n^\beta n^\gamma \nabla_\gamma k_{A\beta} k_A^\alpha . \quad (2.9)$$

We also define  $P_{\perp A} \equiv P_A + u \cdot P_A \hat{k}_A$ , which is orthogonal to both  $k_A$  and  $u$ .

## 2.2 Spacelike separation

**Parallax.** We consider two sources that send bundles of light rays to the observer and label them  $A = 1$  and  $A = 2$ . The observer sees the sources in the directions  $-r_1$  and  $-r_2$ , so the angle  $\varphi$  separating them is given by

$$g \equiv \cos \varphi = r_1 \cdot r_2 = \hat{k}_1 \cdot \hat{k}_2 + 1 . \quad (2.10)$$

Following [23], we define parallax as the change of  $g$ . According to this definition, rigid rotation of the celestial sphere relative to a local inertial frame, as in the Gödel universe, is not parallax. The case of classic parallax with one source and spacelike separation between observation points, as well as our case with two sources, for both spacelike and timelike separation, is illustrated in figure 2.

We first consider observation points separated by a spacelike interval. We denote the spacelike unit vector in the direction of separation by  $s$ , so  $s \cdot s = 1$ , and we also assume that  $s \cdot n = 0$ .<sup>4</sup> We obtain

$$\begin{aligned} g' &\equiv s^\alpha \nabla_\alpha g \\ &= E_1^{(u)-1} \hat{k}_2^\beta s^\alpha \nabla_\alpha k_{1\beta} + \left( E_1^{(u)-1} u^\beta s^\alpha \nabla_\alpha k_{1\beta} + \hat{k}_1^\beta s^\alpha \nabla_\alpha u_\beta \right) \hat{k}_1 \cdot \hat{k}_2 + (1 \leftrightarrow 2) \\ &\simeq \frac{1}{2} r_1 \cdot s P_{\perp 1} \cdot r_2 + (r_2 \cdot s - r_1 \cdot r_2 r_1 \cdot s) \left( \frac{1}{2} E_1^{(n)-1} \tilde{\theta}_1 - \frac{1}{3} \theta^{(u)} \right) + E_1^{(n)-1} \tilde{\sigma}_{1\alpha\beta} r_2^\alpha s^\beta \\ &\quad - (1 - r_1 \cdot r_2) \left( \sigma_{\alpha\beta}^{(u)} r_1^\alpha s^\beta + \omega_{\alpha\beta}^{(u)} r_1^\alpha s^\beta \right) + (1 \leftrightarrow 2) , \end{aligned} \quad (2.11)$$

where we have used the decompositions (2.3) and (2.8),  $(1 \leftrightarrow 2)$  refers to the same expression as the one written down, but with the labels 1 and 2 interchanged, and in the last equality we have taken the limit  $|v| \ll 1$  and dropped all terms that contain any factors of  $v$  without derivatives. Even if  $|v| \ll 1$ , it is possible that  $\nabla_\beta v_\alpha \sim \nabla_\beta n_\alpha$ , so the expansion rate  $\theta^{(u)}$  measured by the observer in the  $u$  frame can be very different from the expansion rate in the  $n$  frame. This is the case for realistic observers located in gravitationally bound structures, where the difference in the velocity with respect to the frame of statistical homogeneity and isotropy is non-relativistic, but the relative difference in the expansion rate is unity.

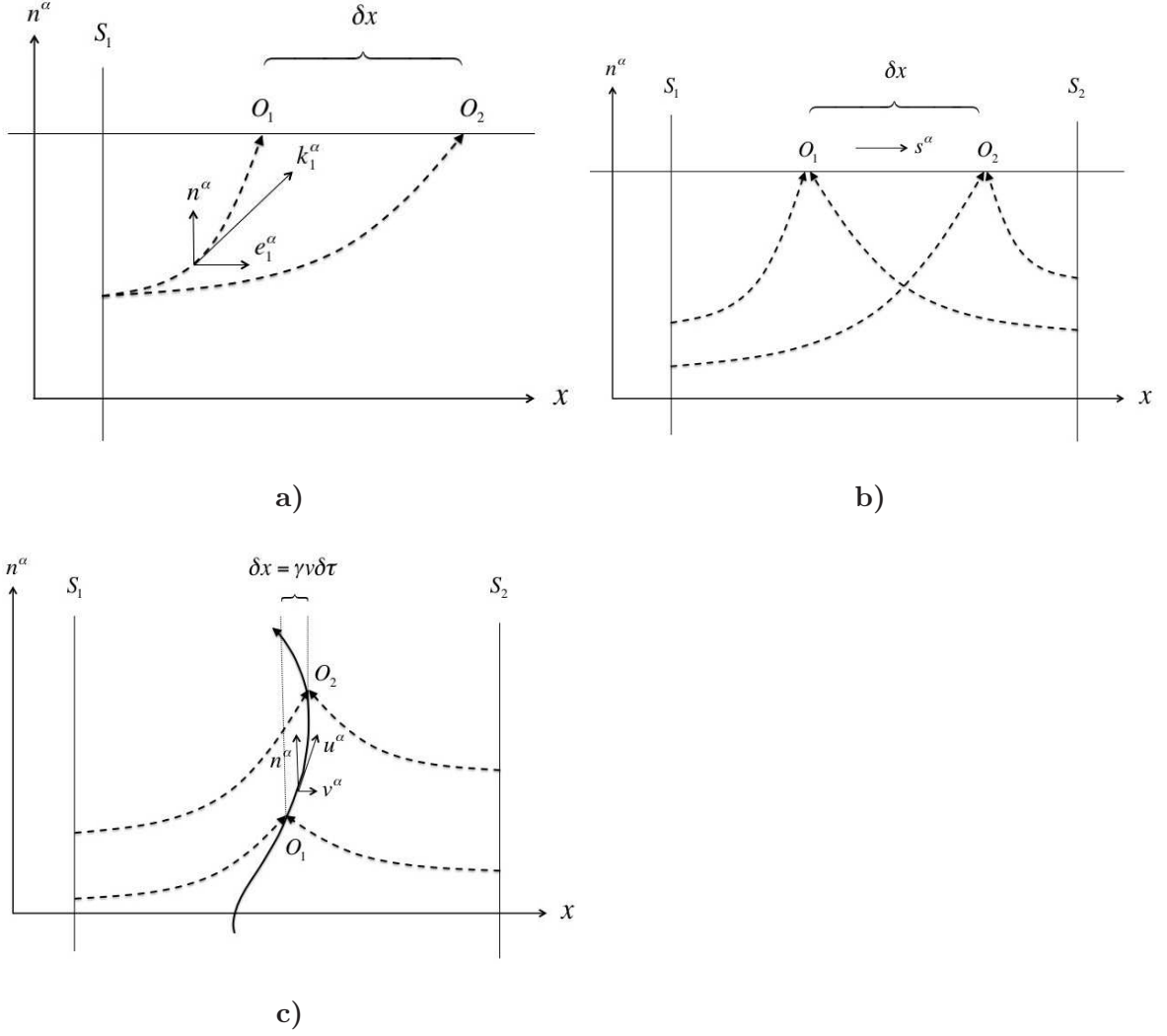
If the observer is displaced a small distance  $\delta x$  in the direction  $s$ , the change in  $g$  is  $\delta g = g' \delta x$ , so the change in the angle is  $\delta \varphi = -(\sin \varphi)^{-1} g' \delta x$ . We thus have

$$\frac{\delta \varphi}{\delta x} = -\frac{1}{\sin \varphi} g' = -\frac{1}{\sqrt{1 - (r_1 \cdot r_2)^2}} g' , \quad (2.12)$$

where all quantities are evaluated at the position of the observer.

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<sup>4</sup>Were  $s$  to contain a component parallel to  $n$ , we would get a linear combination of (2.11) and the  $v$ -independent part of (2.15).



**Figure 2.** Schematic illustration of the different parallax cases. a) Classic parallax. One source  $S_1$  is observed at points  $O_1$  and  $O_2$  separated by the spacelike interval  $\delta x$  on the hypersurface orthogonal to  $n$ . b) Two sources  $S_A$  are observed at points  $O_1$  and  $O_2$  separated by the spacelike interval  $\delta x$  in the direction  $s$  on the hypersurface orthogonal to  $n$ . c) Two sources  $S_A$  are observed at points  $O_1$  and  $O_2$  separated by the timelike interval  $\delta\tau$  in the direction  $u$ . The projected spatial distance between the observation points on the hypersurface orthogonal to  $n$  is  $\delta x = \gamma v \delta\tau$ .

**Comparison to the classic case.** The result (2.11), (2.12) is different from the classic parallax result for spacelike separation [20–22]. The reason for the difference is most transparent when comparing to the analysis of [22], where the light ray from the second observation point is transported to the first observation point. We transport the observer instead, and as a result pick up terms involving the gradient of  $u$  in addition to the gradient of  $k_A$ . (There is also an overall sign difference relative to [21, 22], because we consider future-oriented rays instead of past-oriented rays.) The classic parallax involves the change of angular position of one source with respect to a constant baseline direction. We can get closer to that setup by choosing  $-r_2$  to point in the direction of the source,  $-r_2 = s$ . The term involving  $\hat{\theta}_2$  in (2.11) then vanishes. In classic considerations of parallax, the baseline is also (to zeroth



approximation) orthogonal to the line of sight, so we take  $r_1 \cdot s = 0$ . We then have

$$\begin{aligned} \frac{\delta\varphi}{\delta x} \simeq & \frac{1}{2}P_{\perp 2} \cdot r_1 + \frac{1}{2}E_1^{(n)-1}\tilde{\theta}_1 - \frac{1}{3}\theta^{(u)} + E_1^{(n)-1}\tilde{\sigma}_{1\alpha\beta}s^\alpha s^\beta - E_2^{(n)-1}\tilde{\sigma}_{2\alpha\beta}r_1^\alpha s^\beta \\ & + \sigma_{\alpha\beta}^{(u)}r_1^\alpha s^\beta + \omega_{\alpha\beta}^{(u)}r_1^\alpha s^\beta - \sigma_{\alpha\beta}^{(u)}s^\alpha s^\beta . \end{aligned} \quad (2.13)$$

We can get even closer to the classic analysis, where the baseline is considered to be constant, by neglecting  $P_2$  and  $\tilde{\sigma}_{2\alpha\beta}$ , which describe changes in null geodesic bundle 2 related to the baseline direction, to obtain

$$\frac{\delta\varphi}{\delta x} \simeq \frac{1}{2}E_1^{(n)-1}\tilde{\theta}_1 + E_1^{(n)-1}\tilde{\sigma}_{1\alpha\beta}s^\alpha s^\beta - \frac{1}{3}\theta^{(u)} + \sigma_{\alpha\beta}^{(u)}r_1^\alpha s^\beta + \omega_{\alpha\beta}^{(u)}r_1^\alpha s^\beta - \sigma_{\alpha\beta}^{(u)}s^\alpha s^\beta . \quad (2.14)$$

If we consider an observer whose velocity field has vanishing expansion rate, shear and vorticity, only the first two terms of (2.14) remain, and we recover the classic parallax result [20–22]. However, this does not describe a realistic cosmological situation, because even if the observer is located in a stabilised region with  $\theta^{(u)} = 0$ , the shear and vorticity are in general not zero. This could induce significant angular asymmetry in the parallax, as shear and vorticity can be of the same order of magnitude as the average expansion rate, which in turn is generally of the same order of magnitude as  $E_1^{(n)-1}\tilde{\theta}_1$ . (This will be clear when we consider the FRW spacetime in section 3.) However, the situation with spacelike separation is not relevant for real observations of cosmic parallax. Let us now consider the realistic case where the observation points are separated by a timelike interval.

### 2.3 Timelike separation

**Parallax.** The observationally relevant case is the one where the separation between the observation points is timelike, as they are on the worldline of an observer moving along a timeline curve tangent to  $u$ . The rate of change of the angle  $\varphi$  along the curve is given by

$$\begin{aligned} \dot{g} &\equiv u^\alpha \nabla_\alpha g \\ &= E_1^{(u)-1}\hat{k}_2^\beta u^\alpha \nabla_\alpha k_{1\beta} + \left( E_1^{(u)-1}u^\beta u^\alpha \nabla_\alpha k_{1\beta} + \dot{u} \cdot \hat{k}_1 \right) \hat{k}_1 \cdot \hat{k}_2 + (1 \leftrightarrow 2) \\ &= -\frac{1}{2}P_{\perp 1} \cdot \hat{k}_2 + \hat{k}_1 \cdot \hat{k}_2 \dot{u} \cdot \hat{k}_1 + \gamma \hat{k}_2^\alpha v^\beta \left( \frac{1}{2}\tilde{h}_{1\alpha\beta}E_1^{(n)-1}\tilde{\theta}_1 + E_1^{(n)-1}\tilde{\sigma}_{1\alpha\beta} \right) \\ &\quad + \gamma^2 v^\alpha v^\beta E_1^{(u)-1} \left( \frac{1}{2}\tilde{h}_{1\alpha\beta}\tilde{\theta}_1 + \tilde{\sigma}_{1\alpha\beta} \right) \hat{k}_1 \cdot \hat{k}_2 + (1 \leftrightarrow 2) \\ &\simeq -\frac{1}{2}P_{\perp 1} \cdot r_2 - (1 - r_1 \cdot r_2)\dot{u} \cdot r_1 \\ &\quad + (r_2 \cdot v - r_1 \cdot r_2 r_1 \cdot v) \frac{1}{2}E_1^{(n)-1}\tilde{\theta}_1 + E_1^{(n)-1}\tilde{\sigma}_{1\alpha\beta}r_2^\alpha v^\beta + (1 \leftrightarrow 2) , \end{aligned} \quad (2.15)$$

where we have again applied the decomposition (2.8) and in the last equality taken the limit  $|v| \ll 1$  and dropped terms that contain more than one power of  $v$  (in this case there are no derivatives of  $v$ ). All quantities are again evaluated at the location of the observer. We have not expressed  $\dot{u}$  in terms of  $n$  frame quantities, as we will consider geodesic observers,  $\dot{u} = 0$ . Note that we have kept terms proportional to  $v$ , in contrast to the spacelike separation case (2.11), because in the timelike case motion with respect to the  $n$  frame is important.

If  $v = 0$  (this can be thought of as decomposing  $\nabla_\beta k_{A\alpha}$  with respect to the observer velocity), we have

$$\dot{g} = -\frac{1}{2}P_{\perp 1} \cdot r_2 - (1 - r_1 \cdot r_2)\dot{u} \cdot r_1 + (1 \leftrightarrow 2) , \quad (2.16)$$

and the contributions of  $\tilde{\theta}_A$  and  $\tilde{\sigma}_{A\alpha\beta}$  disappear. If we assume that also the sources move along curves tangent to  $n = u$ , all parallaxes vanish for all observers if and only if  $\sigma_{\alpha\beta}^{(u)} = 0$  and  $\nabla_{[\beta}(\dot{u}_{\alpha]} - \frac{1}{3}\theta^{(u)}u_{\alpha]}) = 0$  [23].

Let us now assume that  $v \neq 0$  and find the change of angle relative to the  $n$  frame spatial displacement. We denote the proper time measured by the observer by  $\tau$ . During a small time interval  $\delta\tau$ , the observer has, in the  $n$  frame, moved the spatial distance  $\delta x = \gamma|v|\delta\tau$ , and the angle has changed by  $\delta\varphi = -(\sin\varphi)^{-1}\dot{g}\delta\tau$ . We thus have

$$\frac{\delta\varphi}{\delta x} = -\frac{1}{\sin\varphi} \frac{\dot{g}}{\gamma|v|} = -\frac{1}{\sqrt{1 - (r_1 \cdot r_2)^2}} \frac{\dot{g}}{\gamma|v|} , \quad (2.17)$$

where all quantities are again evaluated at the position of the observer.

**Comparison to the classic case and the spacelike case.** As in the case of spacelike separation, we get closer to the classic situation by considering angular position with respect to a constant baseline, so we take  $-r_2 = \hat{v}$ , with  $\hat{v} \equiv v/|v|$ , and  $r_1 \cdot r_2 = 0$ . We then have from (2.15) and (2.17)

$$\begin{aligned} \frac{\delta\varphi}{\delta x} \simeq \frac{1}{|v|} & \left[ -\frac{1}{2}P_{\perp 1} \cdot \hat{v} + \frac{1}{2}P_{\perp 2} \cdot r_1 + \dot{u} \cdot r_1 - \dot{u} \cdot \hat{v} \right] \\ & + \frac{1}{2}E_1^{(n)-1}\tilde{\theta}_1 + E_1^{(n)-1}\tilde{\sigma}_{1\alpha\beta}\hat{v}^\alpha\hat{v}^\beta - E_2^{(n)-1}\tilde{\sigma}_{2\alpha\beta}r_1^\alpha\hat{v}^\beta , \end{aligned} \quad (2.18)$$

The first line in (2.18) is the parallax that in general remains even in the case  $v = 0$ , and it is enhanced by  $1/|v|$  relative to the terms on the second line, which vanish when  $v = 0$ . This can be understood as dividing the parallax into a contribution due to deviation of  $n$  from conformal stationarity and a contribution due to observer motion with respect to the  $n$  frame. If  $n$  is identified with the frame of statistical homogeneity and isotropy, the former contribution can be called 'intrinsic' and considered to be due to the properties of the spacetime, and the latter can be thought of as being due to observer motion in space. Such a separation is frame-dependent.

For timelike separation, the change of angle  $\delta\varphi$  between two points separated by spatial distance  $\delta x$  is different than in the case where the observer has moved only spatially. This is to be expected, because in the case of timelike separation also the sources have moved, not just the observer. The difference is twofold: in the timelike case, terms related to the local expansion rate, anisotropic shear and vorticity are absent, while in the spacelike case, there are no terms that persist even in the case of zero spatial motion.

If observer motion is geodesic, the time derivative of  $v$  does not enter into (2.15) or (2.17). However, in writing  $\delta x = \gamma|v|\delta\tau$  we have implicitly assumed that the time interval  $\delta\tau$  is short enough that  $v$  does not change appreciably. For realistic observations, this is not quite true, as the velocity with respect to the CMB has a 10% annual modulation due to motion around the Sun, which should be taken into account in a more detailed analysis.

In [13], it was argued that when considering the parallax due to our motion with respect to the frame of statistical homogeneity and isotropy, the expansion term appearing in (2.14)

should be added to the parallax, using the FRW background expansion rate. In other words, we should consider a baseline that expands in time instead of a fixed baseline. In most of the literature, the fixed baseline has been used, with the exception of [2, 3]. From (2.18) we see that the timelike case corresponds to the fixed baseline equation, and no expansion term needs to be added, unlike argued in [13]. However, the classic result gives only part of the parallax, and there are terms both due to the spacetime geometry and observer motion. We will look at the quantitative importance of these terms in the case of perturbed FRW spacetime in section 4, but let us first consider the definition of the parallax distance, and its relation to angular diameter distance, given that the general result (2.15) and (2.17) contains more terms than are present in the classic definition of parallax distance.

## 2.4 Parallax distance and angular diameter distance

**Definition of parallax distance.** The classic parallax distance is defined as  $D_P = (\delta\varphi/\delta x)^{-1}$ , with the observation points separated by a spacelike interval and the baseline kept constant and orthogonal to the direction of observation. If we were to neglect the local expansion rate, shear and vorticity, (2.13) would give the result  $\delta\varphi/\delta x = \frac{1}{2}E^{(n)-1}\tilde{\theta} + E^{(n)-1}\tilde{\sigma}_{\alpha\beta}s^\alpha s^\beta$  (in this subsection, we consider only a single source, so we drop the subscript  $A$ ). This agrees with the classic result [20–22]. However, for real observations there are extra terms, including ones that are non-zero for  $\delta x = 0$ . The classic definition of  $D_P$  also has the shortcoming that when considering pairs of sources, the parallax varies with direction. We prefer to have a simple definition of the parallax distance to a source, one that is determined from  $(\delta\varphi/\delta x)^{-1}$  but is not equal to it. We define, as in [19],

$$D_P^{-1} \equiv \frac{1}{2}E^{(n)-1}\tilde{\theta}, \quad (2.19)$$

where we consider a light bundle that converges at the source, and the expression is evaluated at the observer. We have defined  $D_P$  in the  $n$  frame. As  $\tilde{\theta}$  is frame-independent, in the observer frame we have  $D_P^{(u)} \equiv E^{(u)}/E^{(n)}D_P = \gamma(1 - v \cdot e)D_P$ .

**Relation to angular diameter distance.** An important property of the parallax distance, noted already in [1], is that it is not trivially related to the angular diameter distance  $D_A$ , unlike the luminosity distance  $D_L$ . In general relativity the Etherington relation  $D_L = (1 + z)^2 D_A$  always holds [18, 19], so measuring the luminosity distance as a function of redshift does not provide extra information compared to the angular diameter distance. The relation between the parallax distance and the angular diameter distance is more subtle. Both distances are defined using  $\tilde{\theta}$ , the area expansion rate of the light bundle. However, in the case of  $D_A$ , light rays converge at the observer, whereas for  $D_P$  they converge at the source. The evolution of  $\tilde{\theta}$  is governed by the Sachs equations,

$$\begin{aligned} \frac{d\tilde{\theta}}{d\lambda} + \frac{1}{2}\tilde{\theta}^2 + 2\tilde{\sigma}^2 &= -R_{\alpha\beta}k^\alpha k^\beta \\ \tilde{h}_\alpha{}^\gamma \tilde{h}_\beta{}^\delta \frac{d\tilde{\sigma}_{\gamma\delta}}{d\lambda} + \tilde{\theta}\tilde{\sigma}_{\alpha\beta} &= -k^\mu k^\nu \tilde{h}_\alpha{}^\gamma \tilde{h}_\beta{}^\delta C_{\mu\gamma\nu\delta}, \end{aligned} \quad (2.20)$$

where  $\lambda$  is the affine parameter so that  $\frac{d}{d\lambda} = k^\alpha \nabla_\alpha$ ,  $R_{\alpha\beta}$  is the Ricci tensor and  $C_{\alpha\beta\gamma\delta}$  is the Weyl tensor. For a light ray converging at the observer, the initial condition is  $\tilde{\theta}_0 = -\infty$ , whereas for a light ray converging at the source, it is  $\tilde{\theta}_s = \infty$ ; the subscript 0 refers to quantities evaluated at the observer's location in time and space, and s refers to quantities

evaluated at the source. In the former case, we have  $D_A \propto \exp\left(\frac{1}{2} \int d\lambda \tilde{\theta}\right)$ , whereas in the latter case we have instead  $D \propto \exp\left(\frac{1}{2} \int d\lambda_0 \tilde{\theta}\right)$ , where  $D \equiv (1+z)D_A$  and  $\lambda_0$  is the affine parameter at the observer [19, 21]<sup>5</sup>. The parallax distance is directly related to the local value of  $\tilde{\theta}$ , because it involves the difference between two light rays at the observer's location, whereas  $D$  and  $D_A$  depend on  $\tilde{\theta}$  via an integral.

Let us consider the case with negligible null shear. (In the real universe, the null shear is small for typical light rays [42, 43].) By comparing the two solutions of (2.20) corresponding to different initial conditions, with rays either converging at the source or at the observer, and using the definition (2.19), we get the following relation between  $D_P$  and  $D_A$  [21]:

$$E_0^{(n)-1} \frac{dD_P^{-1}}{d\lambda} = \frac{1}{D_A^2} \quad \Leftrightarrow \quad D_P^{-1} = \int d\lambda E_0^{(n)} \frac{1}{D_A^2} . \quad (2.21)$$

If we instead consider the change with respect to  $\lambda_0$ , we have [21]

$$D_P^{-1} = E_0^{(n)-1} \frac{1}{D} \frac{dD}{d\lambda_0} . \quad (2.22)$$

The relation (2.21) is a general consistency condition akin to the Etherington relation. It relates two distances measured at the observer, and it holds as long geometrical optics is valid (in parallax case, null shear also has to be negligible). Unlike in the case of the Etherington relation, expressing (2.21) in terms of observational quantities requires converting from the affine parameter to the redshift; we discuss this in more detail in the FRW case in section 3. The relation (2.22) involves the difference in the distance  $D$  to the same source as observed from different positions, so measuring it on cosmological scales is not feasible in the foreseeable future.

### 3 FRW spacetime

#### 3.1 Metric, frame and light bundles

Let us consider parallax in the FRW spacetime, with the metric

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{1}{1-KR^2} dR^2 + R^2 d\Omega^2 \right) , \quad (3.1)$$

where the spatial curvature parameter  $K$  is a constant. We normalise the scale factor to unity today,  $a(t_0) = 1$ .

Taking  $n$  to correspond to the frame where the universe looks homogeneous and isotropic, we have for the observer frame

$$\theta^{(u)} \simeq 3H + \partial_i v^i + \frac{1}{2} u^\alpha \nabla_\alpha |v|^2 , \quad (3.2)$$

where  $H \equiv \frac{1}{a} \frac{da}{dt}$ . and we have taken into account  $|v| \ll 1$ . Note that derivatives of  $v$  can nevertheless be comparable to, or even larger than, background quantities. For an observer on a Solar orbit,  $|v| \sim 10^{-3}$ . However,  $|v|$  changes by about 10% on a timescale of one year,

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<sup>5</sup>The distance  $D$  is known as galaxy area distance [19], proper motion distance [3], transverse comoving distance [40], photon count distance [41] and, most commonly, comoving angular diameter distance.

so the last term in (3.2) is  $\sim 10^{-7} \text{ yr}^{-1} \sim 10^3 H_0$ . (The contribution from motion around the center of the Galaxy is much smaller:  $|v|$  changes by order unity on a timescale of  $10^8$  years, so we have  $\sim 10^{-14} \text{ yr}^{-1} \sim 10^{-4} H_0$ .) In the FRW spacetime, such an observer trajectory is non-geodesic, because  $\dot{u} = 0$  implies  $u^\alpha \nabla_\alpha |v|^2 = -2H|v|^2$ . Even if we neglected small-scale motion around the Sun or the center of the Galaxy, this evolution would be problematic, because for an observer in a stabilised region like a galaxy, the expansion rate should be vanish, but (3.2) is not zero, because  $3H$  and  $\partial_i v^i$  in general have different time-dependence. However, if we opt for non-geodesic motion to get the desired timelike curve, there will be fictitious non-gravitational forces, whose effect on the parallax may be different than those of the real gravitational interactions that are responsible for the small-scale motion and zero expansion rate. These problems arise because deviations from homogeneity and isotropy that cause the observer trajectory to deviate from the background are not included in the description. The resolution is to consistently consider deviations from FRW spacetime. We will look at linear perturbations in the next section, though we will see that it is not possible to treat stabilised regions in linear perturbation theory. In any case, as we saw in section 2, the local expansion rate (and the shear and the vorticity) only affect the parallax in the case of spacelike separation, whereas it is the case of timelike separation that is observationally relevant.

For the light bundle quantities we have

$$\frac{1}{2} E_A^{(n)-1} \tilde{\theta}_A = H + \frac{\sqrt{1 - KR^2}}{aR}, \quad \tilde{\sigma}_{A\alpha\beta} = 0, \quad P_A^\alpha = H E_A^{(n)-1} k_A^\alpha, \quad (3.3)$$

where the source is located at  $R = 0$ ; recall that  $D_P^{-1} = \frac{1}{2} E_{A0}^{(n)-1} \tilde{\theta}_{A0}$ . From  $P_A || k_A$  it follows that  $P_{\perp A} = 0$ . Before discussing  $D_P$ , let us consider parallax in the cases of spacelike and timelike separation.

### 3.2 Spacelike separation

For observation points separated by a spacelike interval, we obtain from (2.11), (2.12) and (3.3)

$$\begin{aligned} \frac{\delta\varphi}{\delta x} \simeq & -\frac{r_2 \cdot s - r_1 \cdot r_2}{\sqrt{1 - (r_1 \cdot r_2)^2}} \left( \frac{1}{2} E_1^{(n)-1} \tilde{\theta}_1 - \frac{1}{3} \theta^{(u)} \right) \\ & + \frac{1 - r_1 \cdot r_2}{\sqrt{1 - (r_1 \cdot r_2)^2}} \left( \sigma_{\alpha\beta}^{(u)} r_1^\alpha s^\beta + \omega_{\alpha\beta}^{(u)} r_1^\alpha s^\beta \right). \end{aligned} \quad (3.4)$$

For an observer comoving with the frame of homogeneity and isotropy ( $v = 0$ ), we have

$$\frac{\delta\varphi}{\delta x} = -\frac{r_2 \cdot s - r_1 \cdot r_2}{\sqrt{1 - (r_1 \cdot r_2)^2}} (D_{P1}^{-1} - H) + (1 \leftrightarrow 2), \quad (3.5)$$

where  $D_{PA}$  is parallax distance to source  $A$ . The parallax between a pair of sources involves the parallax distances to both of them. If we take  $-r_2 = s$  and  $r_1 \cdot r_2 = 0$ , as in section 2, we get the classic parallax result for a single source for expanding baseline,  $\frac{\delta\varphi}{\delta x} = D_{P1}^{-1} - H$  [2, 3]. If we instead consider an observer whose space is not expanding,  $\theta^{(u)} = 0$ , and also put the shear and vorticity to zero, we get the classic result for a fixed baseline,  $\frac{\delta\varphi}{\delta x} = D_{P1}^{-1}$  [1].

### 3.3 Timelike separation

When the observation points are along the worldline of an observer moving with velocity  $u$ , we have from (2.15) and (3.3)

$$\dot{g} \simeq -(1 - r_1 \cdot r_2) \dot{u} \cdot r_1 + (r_2 \cdot v - r_1 \cdot r_2 r_1 \cdot v) D_{P1}^{-1} + (1 \leftrightarrow 2) . \quad (3.6)$$

All terms in (3.6) vanish for  $v = 0$  (as the FRW frame vector  $n$  is geodesic,  $\dot{u}$  goes to zero when  $v \rightarrow 0$ ). For observers whose rest frame coincides with the frame of homogeneity and isotropy, relative angular positions on the sky remain constant in time, as is obvious from the symmetry of the spacetime. However, other observers, with  $v \neq 0$ , measure parallax given by

$$\frac{\delta\varphi}{\delta x} \simeq \frac{1 - r_1 \cdot r_2}{\sqrt{1 - (r_1 \cdot r_2)^2}} \frac{1}{|v|} \dot{u} \cdot r_1 - \frac{r_2 \cdot \hat{v} - r_1 \cdot r_2 r_1 \cdot \hat{v}}{\sqrt{1 - (r_1 \cdot r_2)^2}} D_{P1}^{-1} + (1 \leftrightarrow 2) . \quad (3.7)$$

For a geodesic observer, the first term is zero and we get the same result as in the case of spacelike separation, (3.4), but without the terms related to the expansion rate, shear and vorticity. In the FRW spacetime, observations of parallax give a straightforward measure of the parallax distance. Before considering the effect of deviations from the FRW case, let us introduce a FRW consistency condition involving  $D_P$  and  $D$ .

### 3.4 FRW consistency condition

Let us return to the relation between the parallax distance  $D_P$  and the distance  $D$  (or equivalently  $D_A$  or  $D_L$ ) discussed in section 2.4. From (3.3) we have the parallax distance in the FRW case, taking into account  $a_0 = 1$  and  $R_0 = D$  [1],

$$d_P = \frac{d}{d + \sqrt{1 - kd^2}} , \quad (3.8)$$

where we have defined  $d_P \equiv H_0 D_P$ ,  $d \equiv H_0 D$ ,  $k \equiv K/H_0$ . Note that the parallax distance would asymptote to a finite value even if  $d$  were to grow without limit – though in realistic cosmological models,  $d$  approaches a finite value of order unity at large  $z$ . In other words, the change in the angular position of faraway objects approaches a constant of the order  $H_0 \delta x$  instead of becoming arbitrarily small, as in Euclidean space. This means that the observational difficulty due to the small change in angular separation saturates quickly with increasing redshift, though the problem of observing distant sources remains, as the luminosity distance does grow without limit at large  $z$ .

We can solve for  $k$  from (3.8) to obtain

$$k_P = \frac{1}{d^2} - \left( \frac{1}{d_P} - 1 \right)^2 , \quad (3.9)$$

where we have added the subscript  $P$  to indicate that the spatial curvature constant has been solved from the relation between  $d_P$  and  $d$ . The relation (3.9) provides a test of the FRW metric. If  $d_P$  and  $d$  are measured independently and the combination  $k_P$  is not constant, we can conclude that light propagation in the universe is not described by the FRW metric. (The reverse does not hold true:  $k_P$  may be constant even if light propagation cannot be described by the FRW metric.) Such a test was already pointed out in [1], and it is possible because spatial curvature affects  $d_P$  and  $d$  differently. Note that this test is purely kinematic.

It is based on the geometrical optics treatment of light propagation, and it is independent of the matter content of the universe or the equation of motion that relates the matter to the spacetime geometry.

In the FRW spacetime,  $d$  is related to the expansion rate by

$$d(z) = \frac{1}{\sqrt{-k}} \sinh \left( \sqrt{-k} \int_0^z \frac{d\tilde{z}}{h(\tilde{z})} \right) \quad (3.10)$$

for any value of  $k$ , where we have defined  $h \equiv H/H_0$  and assumed that  $H > 0$  (except possibly at isolated points), and  $z$  is the redshift measured by an observer comoving with matter. From this we can solve for  $k$  to obtain [44]

$$k_H = \frac{1 - h^2 d'^2}{d^2}, \quad (3.11)$$

where the subscript  $H$  refers to the fact that the spatial curvature constant has been solved from the relation between  $h$  and  $d$ , and prime denotes derivative with respect to  $z$ . The relation (3.11) was presented in [44] as way to test of the FRW metric by independently measuring the distance  $d$  and the expansion rate  $h$ .

Combining (3.8) and (3.11) to eliminate  $k$ , we have

$$d_P = \frac{d}{d + h d'} . \quad (3.12)$$

The relation (3.12) is a FRW consistency condition that does not involve the constant  $k$ , but instead relates the three functions  $d_P$ ,  $d$  and  $h$ .

The consistency condition (3.8) can be seen as another test of the FRW relation (3.10) between the distance  $d$  and the expansion rate  $h$ . The relation (2.21) discussed in section 2.4 gives  $d_P(\lambda)$ , if we know  $d_A(\lambda)$ , for any spacetime (assuming zero null shear). However, going from the affine parameter  $\lambda$  to the observable redshift  $z$  involves the expansion rate (and in general also the shear and the acceleration vector). Inserting the FRW result  $E_0^{(n)} d\lambda = -dz/[H(1+z)^2]$  into the general relation (2.21) between  $d_P$  and  $d_A$  and using the FRW result (3.10) between  $d$  and  $h$  gives the FRW relation (3.8) between  $d_P$  and  $d$ , which can also be written as

$$d_P = \frac{1}{1 + \sqrt{-k} \coth \left( \sqrt{-k} \int_0^z \frac{d\tilde{z}}{h(\tilde{z})} \right)} . \quad (3.13)$$

We can treat (3.13) as another consistency condition, involving  $d_P$  and  $h$ . However, (3.9) has the virtue that it involves only the distances  $d$  and  $d_P$ , no integrals or derivatives, unlike (3.11) or (3.13). Also, determining  $d_P$  does not require any information about source properties, whereas finding the expansion rate  $h$  as a function of redshift involves assumptions about the sources, whether using large-scale structure statistics (in particular baryon acoustic oscillations) or galaxy ages.

If light propagation on large scales is not described by the four-dimensional FRW metric, then neither  $k_P$  nor  $k_H$  will be constant in general. Possible deviations that could lead to violation of the FRW relation between  $d$  and  $h$  and thus to non-constant  $k_P$  and  $k_H$  include the universe not being statistically homogeneous and isotropic on large scales [45], the existence of more than three spatial dimensions [46] and the average behaviour on large scales not being FRW despite statistical homogeneity and isotropy, i.e. significant backreaction



[35, 47–52]. But the real universe is in any case not exactly FRW. Before applying the consistency condition (3.9) as a test of whether the FRW metric correctly describes the mean optical properties of spacetime, we should know the corrections due to perturbations around the FRW case.

## 4 Perturbed FRW spacetime

### 4.1 Perturbation theory

Let us consider a general perturbed FRW spacetime. We write the metric as (we follow the notation of [53]; however, we do not assume that the background is spatially flat)

$$ds^2 = a(\eta)^2 \left[ -(1 + 2\phi)d\eta^2 + 2B_i d\eta dx^i + (\gamma_{ij} + 2C_{ij}) dx^i dx^j \right], \quad (4.1)$$

where  $\gamma_{ij}$  is the metric of three-dimensional homogeneous and isotropic space with constant curvature  $6K$ . Indices of the metric perturbations are raised and lowered with  $\gamma_{ij}$ . We denote background quantities by overbar and perturbations by  $\delta$ .

We assume that metric perturbations are  $\mathcal{O}(\epsilon)$ , and that their derivatives with respect to  $t$  are  $\mathcal{O}(H\epsilon)$ . We also assume that spatial derivatives increase the order of magnitude, and that  $\epsilon \ll \partial\epsilon \ll 1$ , where  $\partial$  stands for  $\partial_i/(a|H|)$  (see [54] for a more careful accounting of the smallness conditions). We also assume that  $\delta u^i, |v| \sim \mathcal{O}(\partial\epsilon)$ . We do not assume that terms of order  $\mathcal{O}(\partial^2\epsilon)$  are small.

Under the above assumptions, corrections to the redshift are small, and the average expansion rate is also close to the FRW value [54]. However, corrections to the luminosity distance can be of order unity [55]. If the universe is statistically homogeneous and isotropic, and the distribution evolves slowly compared to the time it takes light to travel the homogeneity scale, then it is expected that the luminosity distance is close to the FRW case, because the average expansion rate is near the FRW value [35, 52] (see also [51]). We are interested in the correction to the parallax  $\frac{\delta\varphi}{\delta x}$  and the parallax distance  $d_P$  (and hence the FRW consistency condition (3.9)).

We only consider the observationally relevant case of timelike separation and geodesic observers. The parallax is then determined in terms of  $P_{\perp A}$ ,  $\hat{\theta}_A$  and  $\tilde{\sigma}_{A\alpha\beta}$  by (2.15) and (2.17). In the exact FRW case,  $P_{\perp A} = 0$ . However, in the perturbed case the intrinsic parallax does not vanish, and even though it is a perturbative contribution, the magnitude can be comparable to the background term due to observer motion, because of the  $1/|v|$  enhancement. Let us first find the magnitude of the intrinsic term and then the term due to motion with respect to the  $n$  frame, which gives the parallax distance.

### 4.2 Perturbed parallax

**The CMB rest frame.** The vector  $P_A$  is determined with respect to the  $n$  frame, implying a split of the observer velocity  $u$  into the velocity of the frame of statistical homogeneity and isotropy  $n$  and the velocity  $v$  with respect to it. The reason for the split is to make more transparent the component of the parallax that is due to our motion with respect to distant sources, which are on average at rest with respect to the frame of statistical homogeneity and isotropy. The total parallax is of course independent of the way we decompose  $u$ . The frame of statistical homogeneity and isotropy is the one in which the matter distribution is statistically homogeneous and isotropic. If primordial perturbations are adiabatic (as they dominantly are [9]), then this is the same as the frame in which the CMB is statistically



isotropic. As the CMB is observed more precisely and analysed more model-independently than galaxy catalogs, it provides a more convenient reference point. The velocity of observers in general differs from the velocity of the frame of statistical homogeneity and isotropy, due to local deviations from exact homogeneity and isotropy, but these variations vanish on average.

The CMB rest frame (also called just the CMB frame) is the frame where the CMB dipole vanishes. As there is a subdominant contribution to the dipole from primordial perturbations and propagation from the last scattering surface to the observer, this frame is different from the frame where the CMB is statistically homogeneous and isotropic. The latter frame is sometimes called the average CMB frame [8]. The frame of statistical isotropy of the CMB can be determined from modulation and aberration of the CMB, though at the moment the constraints are not very precise [8]. Based on inflation, the difference between the frame of statistical homogeneity and isotropy of matter and the CMB rest frame is expected to be at the level of 1%, and the vanishing dipole is a simpler condition. We therefore neglect this difference, and take  $n$  to be the frame of zero CMB dipole.

**Photon momentum and redshift.** Photon momentum from source  $A$  is

$$k_A^\alpha = \bar{k}_A^\alpha + \delta k_A^\alpha = a \bar{k}_A^0 \bar{\tilde{k}}_A^\alpha + \delta k_A^\alpha, \quad (4.2)$$

where we have  $\bar{k}_A^0 = a^{-2} \bar{k}_{A0}^0$  and  $\bar{\tilde{k}}_A^\alpha = (a^{-1}, \bar{e}_A^i) \equiv a^{-1}(1, \bar{e}_A^i)$ , so that  $\gamma_{ij} \bar{e}_A^i \bar{e}_A^j = 1$ . For  $n$  and  $v$  we have, taking into account  $n \cdot n = -1$ ,

$$\begin{aligned} n^\alpha &= \bar{n}^\alpha + \delta n^\alpha = a^{-1} \delta^{\alpha 0} + \delta n^\alpha \simeq (a^{-1}[1 - \phi], n^i) \equiv a^{-1}(1 - \phi, \tilde{n}^i) \\ v^\alpha &= \delta v^\alpha \simeq (0, u^i - n^i). \end{aligned} \quad (4.3)$$

We can solve  $k_A$  from the null geodesic equation  $k_A^\alpha \nabla_\alpha k_A = 0$ . For the time component, we get

$$\frac{\delta k_A^0}{\bar{k}_A^0} \simeq -2\phi + B_i \bar{e}_A^i + \int^\eta d\tilde{\eta} \left( \phi' - B'_i \bar{e}_A^i - C'_{ij} \bar{e}_A^i \bar{e}_A^j \right), \quad (4.4)$$

where we have dropped an integration constant, the integral is along a background null geodesic and prime denotes derivative with respect to  $\eta$ . We see that  $\delta k_A^0 / \bar{k}_A^0 \sim \mathcal{O}(\epsilon)$ .

Redshift in the  $n$  frame is given by  $1 + z_A = E_A^{(n)} / E_{A0}^{(n)}$ , so we have, from (2.4), (4.2), (4.3) and (4.4),

$$1 + z_A \simeq \frac{a_0}{a} \left[ 1 - \phi + \phi_0 - \gamma_{ij} \tilde{n}^i \tilde{e}_A^j + \gamma_{ij} \tilde{n}^i \tilde{e}_A^j|_0 - \int_\eta^{\eta_0} d\tilde{\eta} \left( \phi' - B'_i \bar{e}_A^i - C'_{ij} \bar{e}_A^i \bar{e}_A^j \right) \right]. \quad (4.5)$$

At first sight, it might be tempting to identify the  $\phi$  terms with intrinsic primordial dipole,  $\gamma_{ij} \tilde{n}^i \tilde{e}_A^j$  terms with kinetic dipole and the integral with dipole due to the effects of propagation. This would amount to identifying the hypersurface of statistical homogeneity and isotropy with the hypersurface of constant background time, which is however a gauge dependent quantity. This is clear if we consider the synchronous comoving gauge<sup>6</sup>, where  $\phi = 0, n^i = 0, B_i = 0$ , so that only the integral remains, but obviously the kinetic dipole still exists (see section 5.2 of [54]). The frame of vanishing dipole is defined by  $\int d\Omega e_A^\alpha z_A = 0$ ,

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<sup>6</sup>This gauge restricts the generality of the solution. If the relation between matter and geometry is given by the Einstein equation, it can be chosen for irrotational dust, but not for general matter content.

where  $\int d\Omega$  is an integral over the celestial sphere at the observer, so from (4.5) we get the condition

$$n^k - \frac{3}{4\pi} \int d\Omega e_A^k \bar{g}_{ij} n^i e_A^j|_s = \frac{3}{4\pi} \int d\Omega e_A^k \left[ \phi_s + \int_{\eta_s}^{\eta_0} d\tilde{\eta} \left( \phi' - B'_i \tilde{e}_A^i - C'_{ij} \tilde{e}_A^i \tilde{e}_A^j \right) \right], \quad (4.6)$$

where  $s$  refers to the source (in this case, the last scattering surface). Excepting the possibility that  $n^k$  would be correlated in such a way that there is a large local contribution that cancels against a large contribution from the last scattering surface, we have  $n^k \sim \mathcal{O}(\epsilon)$ .

For the spatial components, it is convenient to rewrite the null geodesic equation as  $k_A^\beta \partial_\beta k_{A\alpha} = \frac{1}{2} \partial_\alpha g_{\mu\nu} k_A^\mu k_A^\nu$  (as in e.g. [56]) to get

$$(\partial_0 + \tilde{e}_A^j \partial_j) \left( \frac{\delta k_{Ai}}{\bar{k}_{A0}^0} \right) = -\partial_i \phi + \tilde{e}_A^j \partial_i B_j + \tilde{e}_A^j \tilde{e}_A^k \partial_i C_{jk} + \frac{\delta k_A^j}{\bar{k}_A^0} \tilde{e}_A^k (\partial_i \gamma_{jk} - \partial_j \gamma_{ik}). \quad (4.7)$$

In the spatially flat case and for Cartesian coordinates, we get (again dropping an integration constant) the simple result

$$\begin{aligned} \frac{\delta k_{Ai}}{\bar{k}_{A0}^0} &= \int^\eta d\tilde{\eta} \left( -\partial_i \phi + \tilde{e}_A^j \partial_i B_j + \tilde{e}_A^j \tilde{e}_A^k \partial_i C_{jk} \right) \\ &= (-\phi + B_j \tilde{e}_A^j + C_{jk} \tilde{e}_A^j \tilde{e}_A^k) \bar{e}^i \\ &\quad + \int^\eta d\tilde{\eta} \left( -\nabla_{A\perp i} \phi + \phi' + \tilde{e}_A^j \nabla_{\perp i} B_j - B'_j \tilde{e}_A^j \tilde{e}_A^i + \tilde{e}_A^j \tilde{e}_A^k \nabla_{\perp i} C_{jk} - C'_{jk} \tilde{e}_A^j \tilde{e}_A^k \tilde{e}_A^i \right) \\ &\simeq \int^\eta d\tilde{\eta} \left( -\nabla_{A\perp i} \phi + \tilde{e}_A^j \nabla_{\perp i} B_j + \tilde{e}_A^j \tilde{e}_A^k \nabla_{\perp i} C_{jk} \right), \end{aligned} \quad (4.8)$$

where  $\nabla_{A\perp i} \equiv (\delta^{ij} - \tilde{e}_A^i \tilde{e}_A^j) \partial_j$ , and in the last equality we have dropped all terms that are definitely not larger than  $\mathcal{O}(\epsilon)$ . The magnitude of the remaining terms could be  $\delta k_{Ai} / \bar{k}_A^0 \sim \mathcal{O}(\partial\epsilon)$ . However, the integral over the derivatives can lead to cancellations that reduce the amplitude.

**Intrinsic contribution.** For  $P_{\perp A}$  we get

$$\begin{aligned} P_{\perp A}^0 &\simeq 0 \\ -\frac{1}{2} a^2 P_{\perp A}^i &\simeq (\delta k_A^i / \bar{k}_A^0)' + \mathcal{H} \tilde{n}^i + \bar{\Gamma}_{jk}^i \tilde{n}^j \tilde{e}_A^k + B^{i'} + \mathcal{H} B^i + \gamma^{ij} \phi_{,j} + \gamma^{ij} (C'_{jk} + B_{[j,k]}) \tilde{e}_A^k \\ &\quad - \left[ (\delta k_A^0 / \bar{k}_A^0)' + \mathcal{H} \gamma_{jk} \tilde{n}^j \tilde{e}_A^k + \phi' + (\phi_{,j} + \mathcal{H} B_j) \tilde{e}_A^j \right] \tilde{e}_A^i, \end{aligned} \quad (4.9)$$

where  $\mathcal{H} \equiv a'/a$ . Corrections to (4.9) are  $\mathcal{O}(H\partial\epsilon\partial\epsilon)$ . The intrinsic term in (2.15) and (2.17) is thus

$$\begin{aligned} &-\frac{1}{2|v|} (P_{\perp 1} \cdot r_2 + P_{\perp 2} \cdot r_1) \\ &\simeq \frac{1}{|v|} \left[ (\delta k_1^i / \bar{k}_1^0)' + \mathcal{H} \tilde{n}^i + \bar{\Gamma}_{jk}^i \tilde{n}^j \tilde{e}_A^k + B^{i'} + \mathcal{H} B^i + \gamma^{ij} \phi_{,j} + \gamma^{ij} C'_{jk} \tilde{e}_1^k \right] \gamma_{il} r_2^l \\ &\quad - \left[ (\delta k_1^0 / \bar{k}_1^0)' + \mathcal{H} \gamma_{jk} \tilde{n}^j \tilde{e}_1^k + \phi' + (\phi_{,j} + \mathcal{H} B_j) \tilde{e}_1^j \right] a^{-1} r_1 \cdot r_2 + (1 \leftrightarrow 2) \\ &\simeq \frac{1}{|v|} \left[ (\delta k_1^i / \bar{k}_1^0)' \gamma_{ij} r_2^j + (1 - r_1 \cdot r_2) \phi_{,j} r_1^j \right] + (1 \leftrightarrow 2), \end{aligned} \quad (4.10)$$

where in the second equality we have dropped subleading terms, taking into account  $n^i \sim \mathcal{O}(\epsilon)$ . The remaining terms are potentially of the same order of magnitude as background quantities. Let us now calculate the contribution to the parallax due to observer motion and see how it compares to (4.10).

**Motion contribution.** The vector  $P_{\perp A}$  involves, to first order, derivatives of  $\delta k_A$  with respect to the background time only. In contrast,  $\tilde{\theta}_A$  and  $\tilde{\sigma}_{A\alpha\beta}$  also involve spatial derivatives of  $\delta k_A$ . Given  $\delta k_A^0/\bar{k}^0 \sim \mathcal{O}(\epsilon)$ ,  $\delta k_A^i/\bar{k}^0 \sim \mathcal{O}(\partial\epsilon)$ , perturbative corrections to motion parallax in (2.15) and (2.17) are potentially  $\mathcal{O}(H\partial^2\epsilon)$ , so they could be as large as the background contribution.

The corrections to  $\tilde{\theta}$  (and thus to  $d_P$ ) and the null shear due to first order perturbations were calculated in [22]<sup>7</sup>, under the assumption that they are small. Evaluating the corrections without this assumption would require numerically integrating the Sachs equations (2.20). Although the source term for the null shear involves the Weyl tensor, which is  $\mathcal{O}(H^2\partial^2\epsilon)$ , the null shear is known to be small in many models of structure [51, 57–59] as well as in the real universe, as determined from observations of image deformation [42, 43]. The reason is that even though the Weyl tensor is locally large, positive and negative contributions along the null geodesic can cancel [35, 57]. The corrections to the parallax due to null shear are thus subdominant to the background contribution.

The source term for  $\tilde{\theta}_A$  involves the Ricci tensor instead of the Weyl tensor. Keeping only the leading terms, we have

$$\begin{aligned} \frac{1}{2}E_A^{(n)-1}\tilde{\theta}_A &\simeq \frac{1}{2}\bar{E}_A^{(n)-1}\bar{\theta}_A + \frac{1}{2}\bar{E}_A^{(n)-1}\partial_i\delta k_A^i \\ &\simeq \bar{D}_{PA}^{-1} + \frac{1}{2}\partial_i \int_{\eta_s}^{\eta} d\tilde{\eta} \left( -\nabla_{A\perp i}\phi + \tilde{e}_A^j \nabla_{\perp i} B_j + \tilde{e}_A^j \tilde{e}_A^k \nabla_{A\perp i} C_{jk} \right), \end{aligned} \quad (4.11)$$

where the second equality holds in the spatially flat case (4.8). The correction could be  $\mathcal{O}(H\partial^2\epsilon)$ , i.e. of the same order of magnitude as the background, or even larger. However, the result depends on the distribution of perturbations, not just on their amplitude. For a statistically homogeneous and isotropic distribution, the magnitude is expected to be suppressed by cancellations in the integral along the null geodesic [35, 52].

**Total parallax.** For the total parallax we get, from (2.15), (2.17), (4.10) and (4.11), dropping subleading terms,

$$\begin{aligned} \frac{\delta\varphi}{\delta x} &\simeq \frac{1}{\sqrt{1-(r_1 \cdot r_2)^2}} \left[ \frac{1}{|v|} \left( (\delta k_1^i/\bar{k}_1^0)' \gamma_{ij} r_2^j + (1 - r_1 \cdot r_2) \phi_{,j} r_1^j \right) \right. \\ &\quad \left. + (r_2 \cdot \hat{v} - r_1 \cdot r_2 \, r_1 \cdot \hat{v}) \left( \bar{D}_{P1}^{-1} + \frac{1}{2}\partial_i(\delta k_1^i/\bar{k}_1^0) \right) \right] + (1 \leftrightarrow 2), \end{aligned} \quad (4.12)$$

where all terms are again evaluated at the observer, and in the spatially flat case  $\delta k_A^i$  is given by (4.8). We have assumed that the observer motion is geodesic, so we can solve  $u^i$  from the condition  $\dot{u}^i = 0$ , with the result

$$u^i \simeq a^{-2}A^i - a^{-1}B^i - a^{-2} \int^t d\tilde{t} \gamma^{ij} \partial_j \phi, \quad (4.13)$$

<sup>7</sup>The contribution of the null shear in [22] corresponds to (2.15) if we take  $-r_2 = v$ . For a general  $r_2$ , the calculation is more involved.

where  $A^i$  does not depend on  $t$  and is related to the initial conditions. If  $B^i = 0$ , we neglect  $A^i$  and assume that we can write  $\phi = b(t)f(x^i)$  (as is the case in linear theory when the decaying mode can be dropped)<sup>8</sup>, taking into account  $v^i \simeq u^i$  we get  $|v| \simeq \frac{\int^t dt b}{ab} \sqrt{\gamma^{jk} \partial_j \phi \partial_k \phi}$  and  $\hat{v}^i = -\frac{a^{-1} \gamma^{ij} \partial_j \phi}{\sqrt{\gamma^{kl} \partial_k \phi \partial_l \phi}}$ . If we also assume that the  $\delta k_A^i$  terms are subdominant, (4.12) reduces to

$$\frac{\delta \varphi}{\delta x} \simeq \frac{1}{\sqrt{1 - (r_1 \cdot r_2)^2}} (r_2 \cdot \hat{v} - r_1 \cdot r_2 \cdot r_1 \cdot \hat{v}) \left( \bar{D}_{P1}^{-1} - \frac{b_0}{\int^{t_0} dt b} \right) + (1 \leftrightarrow 2). \quad (4.14)$$

The intrinsic parallax is of the same order as the motion parallax. It contributes in the same way as the local expansion rate in a FRW universe with spacelike separation between observation points and an observer whose frame is expanding with the background, discussed in (3.2), but with a different numerical coefficient. For example, if we assume that the Einstein equation holds, the matter is dust and the spatial curvature is zero, we have  $b(t) = 1$  and  $\frac{b_0}{\int^{t_0} dt b} = t_0^{-1} = \frac{3}{2} H_0$ , compared to  $H_0$  for the expanding baseline case. In principle, it would be possible to measure the time-dependence of the gravitational potential from the intrinsic parallax, but as it is sensitive to the local gravitational potential, the linear theory prediction cannot be trusted.

**Smallness of the parallax.** The reason that the intrinsic term and the motion term are of the same order of magnitude in (4.14) is that the observer's deviation from the frame of homogeneity and isotropy is generated by the same perturbations that cause the intrinsic parallax term, and both are  $\mathcal{O}(\partial \epsilon)$ . With the parallax distance defined as  $D_{PA}^{-1} = \frac{1}{2} E_A^{(n)-1} \tilde{\theta}_A$ , determining  $D_{PA}$  from parallax measurements requires disentangling the intrinsic term and the motion term, not just measuring the amplitude of the parallax.

In section 1 we noted that in the FRW universe, both CMB anisotropy and parallax vanish for observers comoving with the frame of homogeneity and isotropy. The CMB result is stable in the direction that if the spacetime is close to FRW, the anisotropy is small. However, small CMB anisotropy does not imply that the spacetime is close to FRW [27–30]. The parallax is a dimensional quantity, so it has to be compared to some scale (this is also true when discussing the CMB in the non-FRW case, as derivatives of the temperature anisotropy are involved [27, 29, 30]). In the cosmological case, the relevant scale is given by the Hubble parameter, and for time intervals much smaller than the Hubble time, the parallax is small,  $\delta \varphi \sim H_0 |v| \delta \tau$ . The reverse result would mean that small parallax implies that the universe is close to FRW (or, more generally, close to being conformally stationary). This would mean that the terms in (2.15) are always of the order of the Hubble scale or larger if the spacetime is not close to FRW, which seems unlikely. Indeed, studies of parallax in models with large inhomogeneities or anisotropies [14, 15, 31–33] provide counterexamples (though studies of spherical inhomogeneities have concentrated on locations close to the center, and it is not obvious that the metrics considered cannot be written in terms of a linearly perturbed FRW metric). Note that the smallness of  $P_{A\perp}$  means that  $n^\alpha \nabla_\alpha k_A$  is almost parallel to  $k_A$ ; it would be interesting to study this constraint further.

Closeness to the FRW case can also be considered in terms of the FRW consistency condition introduced in section 3.4. The magnitude of the corrections to the consistency

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<sup>8</sup>As mentioned in section 3.1, linear perturbation theory cannot correctly describe an observer velocity corresponding to motion around the Solar system or the center of the Galaxy: the time evolution of  $u^i$  (and thus  $v^i$ ) is the same everywhere, no complicated local motions are possible.

condition depends on the integral term appearing in  $\delta k_A^i$ , see (4.7) and (4.8). If it is much smaller than background quantities, the FRW consistency condition holds well.

## 5 Discussion

**Non-perturbative deviations from the FRW case.** We have only discussed perturbative deviations from the FRW case. However, even if the universe is statistically homogeneous and isotropic, it is possible that the mean large-scale optical properties of the universe are not well described by the FRW metric, or that the evolution of the average expansion rate over large volumes is not well described by the FRW equations [47–49, 60]. In either case, the metric is not perturbatively close to a single FRW metric everywhere [54]. Another issue that has to be taken into account is that even if the metric at our location is perturbatively close to the FRW metric, it does not satisfy the linearly perturbed FRW equations, as discussed in section 3.1 [61]. So a description beyond linear order, and perhaps beyond perturbation theory, has to be adopted to properly describe observers inside stabilised regions, particularly in the case of complex local motion, such as orbiting the Sun and the center of the Galaxy.

The first study to consider, in connection with the parallax, the possibility that the mean optical properties of the universe are not described by the FRW model was done in the 1970s, using the Dyer–Roeder prescription (which was developed by Zel’dovich) [5, 6]. In the Dyer–Roeder approach, the spacetime dynamics are assumed to be the same as in the FRW model, but the energy density along the null geodesics is taken to be smaller than the FRW density. However, the expansion rate along the null geodesics is taken to be identical to the one in the FRW model. This is inconsistent, and the expansion rate affects the distance via the mapping of the affine parameter to the redshift [35, 43, 52]. It has been argued that for a statistically homogeneous and isotropic distribution the relation between the average expansion rate and the affine parameter is the same as in the FRW model, even though the average expansion rate may be different from the FRW case [35, 51, 52, 62]. It is also expected that  $\theta$ , which gives the parallax distance, is related to the average expansion rate and average density [52]. In this case, the consistency condition (3.9) between  $d_P$  and  $d$  reduces to testing the relation (3.10) between  $h$  and  $d$ : if  $k_H$  is constant, this implies that  $k_P$  is also constant and equal to  $k_P$ . In general, if the average expansion rate is not close to the FRW case,  $k_H$  and  $k_P$  are expected to be different and vary with redshift [50, 51].

**Testing the FRW metric vs. testing homogeneity and isotropy.** The FRW consistency condition (3.9) introduced in section 3.4 is a simple but powerful test. If  $d$  and  $d_P$  are measured at any two redshifts (that are sufficiently large compared to the homogeneity scale) and the corresponding values of  $k_P$  are not equal, the universe on large scales is not described by the four-dimensional FRW metric. More precisely, (3.9) tests the optical properties of the universe, no assumptions have been made about the matter content or its relation to the spacetime geometry. The only assumptions are that the spacetime manifold is 3+1-dimensional and spatially homogeneous and isotropic, and that light propagation can be approximated by geometrical optics. The consistency condition (3.9) tests exact homogeneity and isotropy. We have shown that, apart possibly from the integral term in (4.7) and (4.8), this test is stable to linear perturbations. However, if the mean optical properties of the universe are not well described by the FRW metric, the consistency condition will in general be violated, even if the universe is statistically homogeneous and isotropic.

Several tests of homogeneity and isotropy have been proposed [30, 44, 63–65]. Many of them test statistical homogeneity and isotropy, though this is sometimes conflated with

exact homogeneity and isotropy [63]. Others test for specific deviations from homogeneity and isotropy, such as large spherical inhomogeneities, and are thus particular tests of the Copernican principle (which is a distinct issue from homogeneity and isotropy; see section 1.3 of [49]).

Some proposals test for exact homogeneity and isotropy [25, 65]. However, the universe is not described by the exact FRW metric, and there are locally deviations of order unity in geometrical quantities such as the expansion rate. The relevant questions are then whether the universe is statistically homogeneous and isotropic and whether the metric is perturbatively close to the same FRW model everywhere. These are independent issues. On the one hand, statistical homogeneity and isotropy does not imply that the universe is perturbatively close to FRW, or that light propagation or spatial expansion is well described by the FRW model [51]. On the other hand, without statistical homogeneity, the metric being close to FRW is not a sufficient condition for observables such as the distance  $d$  to be close to the FRW case [55].

Tests of the exact FRW metric are useful if they can be extended to the near-FRW situation. Then they will test either statistical homogeneity and isotropy (like CMB near-anisotropy [29]), or being perturbatively close to the FRW metric. The former are useful for testing models with large spherical inhomogeneities or models with significant global anisotropy. In principle, such tests could also be used to constrain large-scale isocurvature perturbations by comparing the frame of homogeneity and isotropy as inferred from the CMB and from light rays sourced by the matter distribution, though in practice the precision is likely to remain lower than with other methods. At the moment, the only tests that probe whether the universe is close to FRW are the consistency condition (3.9) and the consistency condition (3.11) proposed in [44]. These two consistency conditions have the virtue of using only the metric and geometrical optics. Other such kinematic geometrical optics tests of the FRW metric can be devised and observationally tested.

**Measuring cosmic parallax with Gaia.** The Gaia satellite, launched on December 19, 2013, will probe parallax at cosmological distances. The precision for a single source depends on the magnitude, but for galaxies and quasars at cosmological distances, it is typically of the order  $100 \mu\text{as}$  [10, 12–15]. Assuming that deviations from the FRW model are at most of order unity, the order of magnitude of cosmic parallax is  $\delta\varphi \sim H_0\delta x \sim H_0|v|\delta\tau \sim 10^{-2}\mu\text{as}$   $\delta\tau/\text{yr} \sim 10^{-1}\mu\text{as}$  for  $|v| = 369 \text{ km/s}$  [8] and the Gaia mission duration  $\delta\tau = 5 \text{ yr}$ . The precision for an individual source is therefore three orders of magnitude smaller than the expected cosmic parallax, and parallax distance to individual objects can be measured only for distances of up to about Mpc.

However, Gaia is expected to measure  $N = 5 \times 10^5$  quasars, and measurement error goes down by a factor of  $1/\sqrt{2N} \sim 10^{-3}$ , bringing the precision close to the expected cosmic signal [13–15]. It is not clear how well it will be possible to measure the redshift dependence of  $d_P$ , as the quasars are spread over a large range of redshifts up to  $z = 5$ , the number peaking between  $z = 1$  and  $z = 2$  [11, 16, 17]. As noted in section 3.4,  $d_P(z)$  saturates at large  $z$ . For example, in the spatially flat  $\Lambda\text{CDM}$  model with  $\Omega_{m0} = 0.3$  we have  $d_P(5)/d_P(1) \approx 1.5$ , so parallax remains at roughly the same order of magnitude for all large redshifts probed by Gaia, and increase in measurement difficulty is mostly due to decreasing luminosity. See [13, 14] for the expected precision as a function of redshift, taking the magnitude distribution of quasars into account. In addition to quasars, Gaia is expected to measure the redshifts of  $3 \times 10^6$  galaxies up to  $z = 0.75$  [16], which have not been included in analysis of cosmic



parallax so far.

For testing the consistency condition (3.9), only those redshifts are useful for which there are also measurements of  $d$ . Current observations of type Ia supernovae measure  $d_L = (1+z)d$  only up to  $z = 1.4$  [66], and  $d_A = (1+z)^{-1}d$  has been measured up to  $z = 2.4$  using baryon acoustic oscillations [67], though the analysis is more model-dependent than in the case of supernovae. As the test (3.9) does not involve derivatives or integrals, the measurement of  $d$  needs to be less precise than for the consistency condition (3.11) between  $d$  and  $h$ , though measuring  $d_P$  is more difficult than measuring  $h$ . Constraints on  $k_H$  are at the moment of order  $|k_H| \lesssim 1$  [68, 69], though they are likely to improve perhaps by an order of magnitude when all current and near-future data is included in the analysis.

In Gaia analyses, some faraway sources are assumed to have no parallax (except due to our motion around the center of the Galaxy) and are used to establish a cosmic rest frame [10, 11, 13, 14, 16]. This is potentially problematic from the point of view of measuring cosmic parallax, because in a perturbed FRW spacetime the motion parallax of all sources at cosmological distances is of the same order of magnitude, and this is also the order of magnitude of the intrinsic parallax.

We have not considered small-scale motion around the Sun and the center of the Galaxy. In the literature, the effect of the small-scale velocity and acceleration on the parallax has been studied with Newtonian physics [10, 11, 70, 71], but it would be interesting to include it in a consistent covariant relativistic treatment. Treatment of the local environment is particularly important, as the intrinsic parallax depends on quantities at the location of the observer. The motion of the sources with respect to the frame of statistical homogeneity and isotropy should also be taken into account. As measurements on cosmological distances depend on a large number of sources, this is expected to only add random noise, which can be distinguished from the cosmological signal [13, 14].

The actual precision with which Gaia will measure, or put limits on, the cosmic parallax will depend on the real source distribution (as opposed to the simulated catalogue) and the actual performance of the instrument. The theoretical analysis also needs to be extended to address issues discussed here. Nevertheless, it seems feasible that Gaia could provide a measurement of  $d_P$  at cosmological distances with an accuracy of at least order unity, leading to a test of the FRW metric at a precision comparable to present tests of  $k_H$ .

## 6 Conclusion

**Summary and outlook.** The Gaia satellite offers the possibility of measuring parallax over cosmological distances for the first time in the next few years. Classic discussions of cosmic parallax have considered spacelike separation between observation points and a single source. Using the covariant formalism and considering the angle between a pair of sources, we have calculated the parallax for both spacelike and timelike separation between observation points. We have included both the intrinsic parallax due to the fact that the spacetime is not conformally stationary and the parallax due to observer motion with respect to the mean rest frame of distant sources (which for adiabatic initial conditions is close to the CMB rest frame).

In principle, parallax offers a way to measure cosmological distances without any assumptions about source properties, unlike the angular diameter distance  $d_A$  or the luminosity distance  $d_L$ . In practice, parallax is so small that over cosmological distances it cannot in the near future compete with measurements of angular diameter or luminosity. However, because

the parallax distance contains independent information from  $d_A$  (and  $d_L$ ), their combination makes it possible to test whether the large-scale optical properties of the universe are described by the four-dimensional FRW metric. We have turned this possibility, first mentioned in [1], into a concrete proposal. The test is independent of dynamical issues related to matter content and its relation to the spacetime geometry, it depends only on the validity of the FRW metric and geometrical optics. Such kinematic geometrical optics tests are simple but powerful: a negative result would rule out all four-dimensional FRW models where light propagates as in general relativity. A similar consistency condition between the expansion rate and  $d_A$  (or  $d_L$ ) was introduced in [44], and has been tested observationally [68, 69]. Observations of the Gaia satellite may make it possible to test the parallax consistency condition for the first time.

We have studied the stability of the parallax due to our motion with respect to the frame of homogeneity and isotropy to linear perturbations around the FRW metric. There is one integral term that may potentially be of the same order of magnitude as the background contribution, and the intrinsic parallax due to the perturbations is of the same order of magnitude as the parallax due to our motion. Therefore, when analysing observations, perturbations have to be taken into account. The intrinsic term may be distinguished by its different dependence on redshift.

The perturbative integral correction to the parallax should be evaluated. It should also be taken into account that the local environment of observers located in a stabilised region and undergoing orbital motions cannot be described by the linear equations, but has to be considered with non-linear perturbation theory, or a non-perturbative treatment. The capability of the Gaia satellite to measure cosmic parallax should be analysed in more detail, possibly by combining galaxy and quasar data. In particular, the procedure that some quasars are taken to define a cosmic rest frame by having vanishing intrinsic parallax should be carefully considered.

In addition to the consistency condition between parallax distance and  $d_A$  (or  $d_L$ ) proposed here and the condition between the expansion rate and  $d_A$  (or  $d_L$ ) proposed in [44], it is possible to develop other kinematic tests of the FRW metric based on geometrical optics.

## Acknowledgments

I thank Stanislav Rusak for translating [4].

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